CLT Proof

STAT 510

Thm CENTRAL LIMIT THEOREM (CLT)

"CLASSICAL"

$X_1, X_2, \ldots X_n$ IID $\mathbb{E}[X_i] = \mu$, $\mathbb{V}[X_i] = \sigma^2 < \infty$

DEFINE $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$

THEN

$$Z_n = \frac{(\bar{X}_n - \mu)}{\sqrt{\mathbb{V}[\bar{X}_n]}} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z$$

$Z \sim N(0,1)$

RECALL

RV X HAS MGF $\psi_X(t) = \mathbb{E}\left[e^{tX}\right]$

ASSUME $\psi(t)$ FINITE IN NEIGHBORHOOD AROUND t

LEMMA

$Z_1, Z_2, Z_3, \ldots$ RVS. $\psi_n(t)$ MGF OF Z_n

$\psi_Z(t)$ MGF OF Z

IF $\psi_n(t) \longrightarrow \psi_Z(t)$ FOR ALL t IN OPEN INTERVAL AROUND 0

THEN $Z_n \xrightarrow{d} Z$.

MGF PROPERTIES

1. $Y = aX + b \implies \psi_Y(t) = e^{bt}\psi_X(at)$

2. $X_1 \ldots X_n$ IND AND $Y = \sum_{i=1}^{n} X_i$

$$\implies \psi_Y(t) = \prod_{i=1}^{n} \psi_{X_i}(t)$$

WHERE $\psi_{X_i}(t)$ IS THE MGF OF X_i

THM X, Y RVs

IF $\varphi_X(t) = \varphi_Y(t)$ FOR ALL t IN AN OPEN INTERVAL AROUND 0

THEN $X \overset{d}{=} Y$.

RECALL, DOES NOT MEAN $X = Y$.

PROOF

DEFINE $Y_i = \frac{X_i - \mu}{\sigma}$

$\mathbb{E}[Y_i] = 0$

$\mathbb{V}[Y_i] = 1$

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$$

LET $\psi(t)$ BE MGF OF Y_i.

RECALL X_i IID

SO Y_i IID

MGF OF $\sum_{i=1}^{n} Y_i$ IS $\left[\psi(t)\right]^n$.

MGF OF Z_n IS $\left[\psi(t/\sqrt{n})\right]^n \equiv \xi_n(t)$

NOTE THAT

$$\psi'(0) = \mathbb{E}[Y_i] = 0$$

$$\psi''(0) = \mathbb{E}[Y_i^2] = \mathbb{V}[Y_i] = 1$$

$$\psi(t) = \psi(0) + t\,\psi'(0) + \frac{t^2}{2!}\psi''(0) + \frac{t^3}{3!}\psi'''(0) + \dots$$

$$= 1 + 0 + \frac{t^2}{2!} + \frac{t^3}{3!}\psi'''(0) + \dots$$

$$\xi_n(t) = \left[\psi\left(t/\sqrt{n}\right)\right]^n$$

$$= \left[1 + \frac{t^2}{2n} + \frac{t^3}{3!\,n^{3/2}}\psi'''(0) + \dots\right]^n$$

MGF OF $N(0,1)$

$$= \left[1 + \frac{\frac{t^2}{2} + \frac{t^3}{3!\,n^{1/2}}\psi'''(0) + \dots}{n}\right]^n \longrightarrow e^{t^2/2}$$

IF $a_n \to a$ THEN $\left(1 + \frac{a_n}{n}\right)^n \to e^a$

Thus

$$Z_n \xrightarrow{d} Z$$