

CLT PROOF

STAT 510

Thm

CENTRAL LIMIT THEOREM (CLT)
"CLASSICAL"

$$X_1, X_2, \dots, X_n \text{ i.i.d. } E[X_i] = \mu, \text{ } V[X_i] = \sigma^2 < \infty$$

DEFINE $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$

THEN

$$Z_n = \frac{(\bar{X}_n - \mu)}{\sqrt{V[\bar{X}_n]}} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z$$

$$Z \sim N(0, 1)$$

RECALL

R.V. X HAS MGF $\psi_X(t) = \mathbb{E}[e^{tX}]$

ASSUME $\psi(t)$ FINITE IN
NEIGHBORHOOD AROUND t

LEMMA

Z_1, Z_2, Z_3, \dots R.V.s. $\psi_n(t)$ MGF OF Z_n

$\psi_Z(t)$ MGF OF Z

IF $\psi_n(t) \rightarrow \psi_Z(t)$ FOR ALL t IN OPEN INTERVAL
AROUND 0

THEN $Z_n \xrightarrow{d} Z$.

MGF PROPERTIES

$$(1) \quad Y = aX + b \implies \psi_Y(t) = e^{bt} \psi_X(at)$$

$$(2) \quad X_1, \dots, X_n \text{ IND AND } Y = \sum_{i=1}^n X_i$$

$$\implies \psi_Y(t) = \prod_{i=1}^n \psi_{X_i}(t)$$

WHERE $\psi_{X_i}(t)$ IS THE MGF OF X_i

Thm

X, Y RVs

$$\text{IF } \psi_X(t) = \psi_Y(t)$$

FOR ALL t IN AN
OPEN INTERVAL AROUND 0

THEN $X \stackrel{d}{=} Y$.

RECALL, DOES NOT MEAN $X = Y$.

PROOF

DEFINE $Y_i = \frac{X_i - \mu}{\sigma}$

$$E[Y_i] = 0$$

$$\forall [Y_i] = 1$$

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i$$

LET $\psi(t)$ BE MGF OF Y_i .

RECALL X_i IID
SO Y_i IID

MGF OF $\sum_{i=1}^n Y_i$ IS $[\psi(t)]^n$.

MGF OF Z_n IS $[\psi(t/\sqrt{n})]^n \equiv \tilde{\xi}_n(t)$

NOTE THAT

$$\psi'(0) = E[Y_i] = 0$$

$$\psi''(0) = E[Y_i^2] = \text{Var}[Y_i] = 1$$

$$\psi(t) = \psi(0) + t \psi'(0) + \frac{t^2}{2!} \psi''(0) + \frac{t^3}{3!} \psi'''(0) + \dots$$

$$= 1 + 0 + \frac{t^2}{2!} + \frac{t^3}{3!} \psi'''(0) + \dots$$

$$\xi_n(t) = \left[\psi\left(\frac{t}{\sqrt{n}}\right) \right]^n$$

$$= \left[1 + \frac{t^2}{2n} + \frac{t^3}{3! n^{3/2}} \psi'''(0) + \dots \right]^n$$

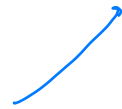
$$= \left[1 + \frac{\frac{t^2}{2} + \frac{t^3}{3! n^{3/2}} \psi'''(0) + \dots}{1} \right]^n$$

MGF OF $N(0,1)$



$$e^{t^2/2}$$

IF $a_n \rightarrow a$ THEN $\left(1 + \frac{a_n}{n}\right)^n \rightarrow e^a$



Thus

$$Z_1 \xrightarrow{\partial} Z$$

