INEQUALITIES

STAT 510

THM MARKOV'S INEQUALITY

LET X BE A NON-NEGATIVE RV AND SUPPOSE $\mathbb{E}[X]$ EXISTS

THEN FOR ANY $t > 0$

$$P(X > t) \leq \frac{\mathbb{E}[X]}{t}$$

Proof

Note $X \geq 0$

$$\mathbb{E}[X] = \int_0^\infty x f(x)\,dx = \overbrace{\int_0^t x f(x)\,dx}^{\geq 0} + \int_t^\infty x f(x)\,dx$$

$$\geq \int_t^\infty x f(x)\,dx \geq t \int_t^\infty f(x)\,dx = t \cdot P(X \geq t)$$

$$\Rightarrow \quad P(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

THM CHEBYSHEV'S INEQUALITY

LET $\mu = \mathbb{E}[X]$ AND $\sigma^2 = \mathbb{V}[X]$, THEN

$$P(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

PROOF

MARKOV

$$P(|X-\mu| \geq t) = P(|X-\mu|^2 \geq t^2) \leq \frac{\mathbb{E}[(X-\mu)^2]}{t^2} = \frac{\sigma^2}{t^2}$$

Example

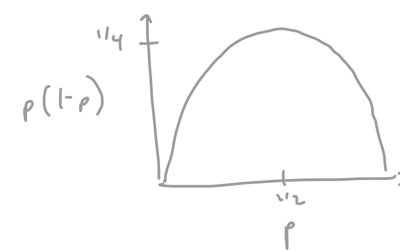
$X_i \sim \text{Bernoulli}(p)$ $\quad \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

How far is $\bar{X}_n$ from p?

Note $\mathbb{E}[X_i] = p$ $\quad \mathbb{V}[X_i] = p(1-p)$

$$P(|\bar{X}_n - p| > \varepsilon) \leq \frac{\mathbb{V}[\bar{X}_n]}{\varepsilon^2} = \frac{p(1-p)}{n\varepsilon^2} \leq \frac{1}{4n\varepsilon^2}$$

Since $p(1-p) \leq 1/4$



Note prob $\to 0$ as $n \to \infty$

THM HOEFFDING'S INEQUALITY

$Y_1, Y_2, \dots Y_n$ IND WITH $\mathbb{E}[Y_i] = 0$

AND $a_i \le Y_i \le b_i$

LET $\varepsilon > 0$. THEN FOR $t > 0$

$$P\left(\sum_{i=1}^{n} Y_i \ge \varepsilon\right) \le e^{-t\varepsilon} \prod_{i=1}^{n} e^{t^2 (b_i - a_i)^2 / 8}$$

MORE CONDITIONS GIVE SHARPER INEQUALITY.

THM

$$X_1, X_2, \ldots X_n \sim \text{Bernoulli}(p)$$

For $\varepsilon > 0$

$$P\left(\left|\overline{X}_n - p\right| > \varepsilon\right) \le 2e^{-2n\varepsilon^2}$$

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Proof via Hoeffding's

- Try yourself!
- In appendix

Example

Chebyshev's $\quad P(|\overline{X}_n - p| > \varepsilon) \leq \dfrac{1}{4n\varepsilon^2} = \dfrac{1}{4(200)(0.1)^2}$

$= 0.125$

Hoeffding's $\quad P(|\overline{X}_n - p| > \varepsilon) \leq 2e^{-2n\varepsilon^2} = 2e^{-2(200)(0.1)^2}$

$= 0.03663\ldots$

When $n = 200$

$\varepsilon = 0.1$

TWO INEQUALITIES FOR EXPECTATIONS

THM CAUCHY - SWARTZ INEQUALITY

GIVEN X, Y WITH FINITE VARIANCE

$$\mathbb{E}[|XY|] \leq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$$

THM JENSEN'S INEQUALITY

IF g CONVEX $\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$

IF g CONCAVE $\mathbb{E}[g(X)] \leq g(\mathbb{E}[X])$