Conditional Expectation

STAT 510

Defn The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X \mid Y = y] = \begin{cases} \sum x f(x \mid y) \, dx & X \text{ disc} \\ \int x f(x \mid y) \, dx & X \text{ cont} \end{cases}$$

Thm Rule of iterated expectation / Adam's law

$$\mathbb{E}\left[\mathbb{E}[Y|X]\right] = \mathbb{E}[Y]$$

and

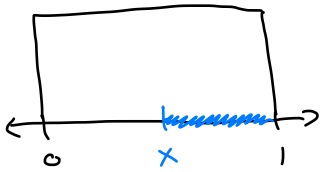
$$\mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}[X]$$

Example

$X \sim \text{UNIF}(0,1)$

$Y \mid X = x \sim \text{UNIF}(x,1)$

$\mathbb{E}[Y] = ?$



$$\mathbb{E}[Y \mid X = x] = \frac{x+1}{2}$$ ← Function of x, returns #

$$\mathbb{E}[Y \mid X] = \frac{X+1}{2}$$ ← Random variable

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}\left[\frac{X+1}{2}\right] = \frac{\mathbb{E}[X]+1}{2}$$

$$= \frac{0.5+1}{2} = 0.75$$

DEFN CONDITIONAL VARIANCE

$$\mathbb{V}\left[Y \mid X = x\right] = \int \left(y - m(x)\right)^2 f(y|x)\, dy$$

WHERE $m(x) = \mathbb{E}\left[Y \mid X = x\right]$

THM VARIANCE DECOMPOSITION / EVE'S LAW

$$\mathbb{V}[Y] = \mathbb{E}[\mathbb{V}[Y|X]] + \mathbb{V}[\mathbb{E}[Y|X]]$$

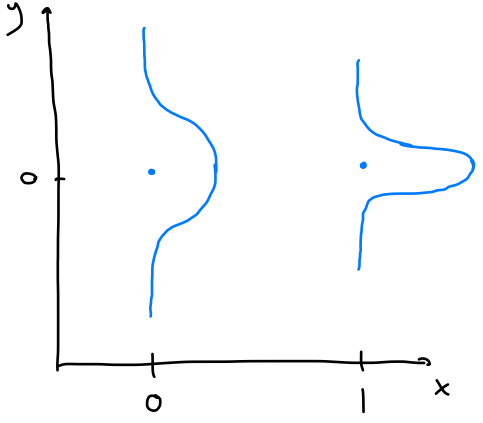
MARGINAL VARIANCE OF Y

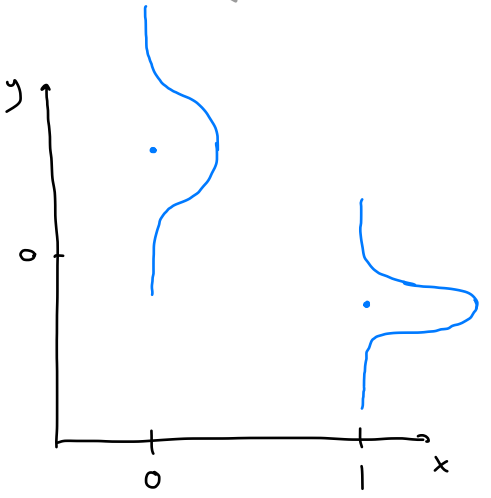
CONDITIONAL VARIANCE OF $Y|X$

CONDITIONAL MEAN OF $Y|X$

$$P[X=0] = P[X=1] = 1/2$$

SAME AVERAGE
CONDITIONAL VARIANCE





MORE TOTAL VARIATION