EXPECTATION

STAT 510

DEFN Expected Value of a random variable X

→ mean

→ first moment

$$\mathbb{E}[X] = \int x \, dF(x) = \begin{cases} \sum_x x f(x) & X \text{ disc} \\ \int x f(x) \, dx & X \text{ cont} \end{cases}$$

PMF

PDF

$$\mathbb{E}[X] = \mathbb{E}X = \int x \, dF(x) = \mu_X = \mu$$

← NOTATION

Example

X	f(x)
0	1/4
1	1/2
2	1/4

$$\mathbb{E}[X] = \sum_{x=0}^{2} x f(x)$$

$P[X=0]$

$$= 0 \cdot f(0) + 1 \cdot f(1) + 2 \cdot f(2)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \boxed{1}$$

Example

$X \sim \text{Unif}(-2, 5)$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{-2}^{5} x \cdot \frac{1}{7}\, dx$$

$$= \boxed{1.5} \neq 1$$

EXAMPLE

FLIP A COIN

IF HEADS, YOU WIN \$100.

IF TAILS, YOU PAY \$5.

DO YOU TAKE THIS BET?

NO!!

ASSUMES FAIR COIN!

	x	$f(x)$	
H →	-5	$1/2$	→ $1-p$
T →	100	$1/2$	→ p

A FAIR BET IF $\mathbb{E}[x] = 0$

IF FAIR COIN → $\mathbb{E}[x] = \sum_x x f(x) = -5(1/2) + 100(1/2) = 47.5$

$$\mathbb{E}[x] = -5(1-p) + 100(p) = 105p - 5 = 5(21p-1)$$

$\implies$ FAIR BET IF $p = 1/21$

EXAMPLE

SAINT PETERSBURG PARADOX

FLIP A FAIR COIN UNTIL YOU SEE TAILS.

YOU WIN 2^k DOLLARS WHERE k = # FLIPS.

$2, 4, 8, 16, \ldots$

HOW MUCH WOULD YOU PAY TO PLAY THIS GAME?

DEFINE X = MONEY WON

UNDEFINED EXPECTATION

$$\mathbb{E}[X] = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + \ldots = \infty$$

ANY FINITE AMOUNT???

SEE ALSO CAUCHY

THM RULE OF THE LAZY STATISTICIAN

LET $Y = r(X)$. THEN,

$$\mathbb{E}[Y] = \mathbb{E}[r(X)] = \int r(x)\, dF(x) = \begin{cases} \sum_x r(x) f(x) & X \text{ DISC} \\ \int r(x) f(x)\, dx & X \text{ CONT} \end{cases}$$

Example

Let $X \sim \text{UNIF}(0,2) \longrightarrow f(x) = \frac{1}{2} \quad 0 < x < 2$

Set $Y = e^X = r(X)$

DEFN

PDF OF Y

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y\, f(y)\, dx$$

$$= \mathbb{E}\left[e^X\right] = \int_0^2 e^x \cdot \frac{1}{2}\, dx = \frac{1}{2}\left[e^x\right]_{x=0}^{x=2}$$

LAZY

$$= \frac{1}{2}\left[e^2 - 1\right]$$

Thm $X_1, X_2, \ldots, X_n$ RVs and $a_1, a_2, \ldots, a_n$ constants

Then $$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i]$$

"Linearity of expectation"

Thm $X_1, X_2, \ldots, X_n$ IND

Then $$\mathbb{E}\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} \mathbb{E}[X_i]$$

Example

$X \sim \text{Binom}(n, p)$

$$\mathbb{E}[X] = \sum_{x=0}^{n} x f(x) = \sum_{x=0}^{\infty} x \binom{n}{x} p^x (1-p)^{n-x} = ???$$

Instead define $X = \sum_{i=1}^{n} X_i \qquad X_i \sim \text{Bern}(p)$

$$\mathbb{E}[X_i] = 0 \cdot (1-p) + 1 \cdot p = p$$

Then $\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = np.$ ✓