STAT 510: Homework 02

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Due: Monday, February 15, 11:59 PM

Exercise 1 (Some Simple Expectations)

Let X denote the number of times Ron Weasley manages to irritate Professor Snape in one day. Then X has the following probability distribution:

x	$f(x)$
0	0.15
1	0.20
2	0.20
3	0.30
4	0.15

- Find the expected number of times Ron will get in trouble with Professor Snape, $\text{E}[X]$.
- Find the standard deviation of the number of times Ron will get in trouble with Professor Snape, $\text{SD}[X]$.
- Each day, Professor Snape takes 20 points from Gryffindor, simply because he can. Additionally, Professor Snape takes 10 points from Gryffindor each time Ron Weasley irritates him. If these are the only two sources of point deductions for Gryffindor, what is the expected point loss for Gryffindor each day?
- What is the standard deviation of points lost for Gryffindor each day?

Exercise 2 (Expectation of a Maximum)

(**LW** 3.3) Let $X_1, \ldots, X_n \sim \text{Uniform}(0, 1)$. Define $Y_n = \max\{X_1, \ldots, X_n\}$. Find $\mathbb{E}[Y_n]$.

Exercise 3 (A Random Walk)

(**LW** 3.4) A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is $1 - p$ that the particle will jump one unit to the right. Let X_n be the position of the particle after n jumps. Find $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$.

Exercise 4 (A Lazy Statistican Does Algebra)

(**LW** 3.10) Let $X \sim \text{Normal}(0, 1)$. Define $Y = e^X$. Find $\mathbb{E}[Y]$ and $\mathbb{V}[Y]$. Your answer should be a function of e, not a decimal representation.

Exercise 5 (A Simple Hierarchical Model)

(**LW** 3.13) Suppose we generate a random variable X in the following way. First we flip a fair coin. If the coin is heads, take X to have a $\text{Uniform}(0, 1)$ distribution. If the coin is tails, take X to have a $\text{Uniform}(3, 4)$ distribution. Find the mean and standard deviation of X.

Exercise 6 (Variance and Covariance)

(**LW** 3.15) Let

$$f_{X,Y}(x, y) = \frac{1}{3}(x + y), \quad 0 < x < 1,\ 0 < y < 2$$

Find $\mathbb{V}[2X - 3Y + 8]$.

Exercise 7 (Checking Independence)

(**LW** 3.22) Let $X \sim \text{Uniform}(0, 1)$. Let $0 < a < b < 1$. Let

$$Y = \begin{cases} 1 & 0 < x < b \\ 0 & \text{otherwise} \end{cases}$$

and let

$$Z = \begin{cases} 1 & a < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Are Y and Z independent? (Why or why not?) Find $\mathbb{E}[Y \mid Z]$.

Exercise 8 (Using Moment Generating Functions)

(**LW** 3.24) Let $X_1, \ldots, X_n \sim \text{Exp}(\beta)$. Find the moment generating function of X_i and use this to show that

$$\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \beta).$$

Exercise 9 (The Classic Setup)

(**LW** 3.8) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with $\mathbb{E}[X_i] = \mu$ and $\mathbb{V}[X_i] = \sigma^2$.

Prove that,

$$\mathbb{E}[\bar{X}_n] = \mu, \quad \mathbb{V}[\bar{X}_n] = \frac{\sigma^2}{n}, \quad \text{and} \quad \mathbb{E}[S_n^2] = \sigma^2.$$

Exercise 10 (Unintuitive Intuitions)

(Based on **LW** 3.9) Let $X_1, \ldots, X_n \sim \text{Normal}(0, 1)$ and $Y_1, \ldots, Y_n \sim \text{Cauchy}$. (In particular a Cauchy distribution with a location parameter of 0 and a scale parameter of 1.) Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

Generate a (single) random sample of size $n = 10,000$ for both distributions and plot $\bar{x}_n$ and $\bar{y}_n$ versus n for $n = 1, \ldots, 10,000$ on a single plot. (Use a different color for each distribution.) Repeat this process three times. Display the plots in a 1×3 grid. Explain why there is such a difference between the two distributions.

Exercise 11 (Simulating a Stock Market)

(Based on **LW** 2.11) Let $Y_1, Y_2, \ldots$ be independent random variables such that

$$P(Y_i = -1) = P(Y_i = 1) = 0.5.$$

Define

$$X_n = \sum_{i=1}^{n} Y_i.$$

Think of $Y_i = 1$ as "the stock price increased by one dollar," $Y_i = -1$ as "the stock price decreased by one dollar," and X_n as the value of the stock on day n. Find $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$.

Simulate X_n and plot X_n versus n for $n = 10,000$. Repeat this process three times. (So, simulate the price of three stocks for $10,000$ days.) Use a single plot with a different color for each stock. Notice two things. First, it is easy to "see" patterns in the sequence even though it is random. Second, you will find that the three runs look very different even though they were generated the same way. How do the expectations you found explain this observation? (Also, consider repeating this process more times than needed to simulate three stocks for more intuition.)

Note: This is an incredibly simplistic model of a market. Please do not make any decisions in the real world based on this model.